NEW GENERALIZATION FRACTIONAL INEQUALITIES OF OSTROWSKI-GRÜSS TYPE

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Abstract. In this paper, we use the Riemann-Liouville fractional integrals to establish some new integral inequalities of Ostrowski-Grüss type. From our results, the classical Ostrowski-Grüss type inequalities can be deduced as some special cases.

1. Introduction

Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) and assume $|f'(x)|\leq M$ for all $x\in(a,b)$. Then the following inequality holds:

$$|S(f;a,b)|\leq\frac{M}{b-a}\left[\left(\frac{b-a}{2}\right)^2+\left(x-\frac{a+b}{2}\right)^2\right] \tag{1.1}$$

for all $x\in[a,b]$ where

$$S(f;a,b)=f(x)-\mathcal{M}\left(f;a,b\right)$$

and

$$\mathcal{M}\left(f;a,b\right)=\frac{1}{b-a}\int_a^b f(x)dx. \tag{1.2}$$

This inequality is well known in the literature as Ostrowski inequality.

In 1882, P. L. Čebyšev [4] gave the following inequality:

$$|T(f,g)|\leq\frac{1}{12}(b-a)^2\left\|f'\right\|_\infty\left\|g'\right\|_\infty, \tag{1.3}$$

where $f,g:[a,b]\to\mathbb{R}$ are absolutely continuous function, whose first derivatives f' and g' are bounded,

$$\begin{aligned} T(f,g) &= \frac{1}{b-a}\int_a^b f(x)g(x)dx-\left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right) \\ &= \mathcal{M}\left(fg;a,b\right)-\mathcal{M}\left(f;a,b\right)\mathcal{M}\left(g;a,b\right) \end{aligned} \tag{1.4}$$

and $\|.\|_\infty$ denotes the norm in $L_\infty[a,b]$ defined as $\|p\|_\infty=ess\sup\limits_{t\in[a,b]}|p(t)|$.

In 1935, G. Grüss [5] proved the following inequality:

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$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx\frac{1}{b-a}\int_a^b g(x)dx\right| \leq \frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma), \tag{1.5}$$

provided that f and g are two integrable function on $[a,b]$ satisfying the condition

$$\varphi \leq f(x) \leq \Phi \quad \text{and} \quad \gamma \leq g(x) \leq \Gamma \text{ for all } x \in [a,b]. \tag{1.6}$$

The constant $\frac{1}{4}$ is best possible.

From [16], if $f:[a,b]\to\mathbb{R}$ is differentiable on $[a,b]$ with the first derivative f' integrable on $[a,b]$, then Montgomery identity holds:

$$f(x) = \frac{1}{b-a}\int_a^b f(t)dt + \int_a^b P_1(x,t)f'(t)dt, \tag{1.7}$$

where $P_1(x,t)$ is the Peano kernel defined by

$$P_1(x,t) = \begin{cases} \dfrac{t-a}{b-a}, & a \leq t < x \\ \dfrac{t-b}{b-a}, & x \leq t \leq b. \end{cases}$$

This inequality provides an upper bound for the approximation of integral mean of a function f by the functional value $f(x)$ at $x\in[a,b]$. In 2001, Cheng [1] proved the following Ostrowski-Grüss type integral inequality.

Theorem 1. *Let $I \subset \mathbb{R}$ be an open interval, $a,b \in I, a<b$. If $f:I\to\mathbb{R}$ is a differentiable function such that there exist constants $\gamma,\Gamma\in\mathbb{R}$, with $\varphi \leq f^{'}(x) \leq \Phi$, $x\in[a,b]$. Then have*

$$\left|f(x) - \frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{1.8}$$
$$\leq \frac{1}{4}(b-a)(\Phi-\varphi)\,, \text{ for all } x\in[a,b]\,.$$

In [6], Matic et. al gave the following theorem by use of Grüss inequality:

Theorem 2. *Let the assumptions of Theorem 1 hold. Then for all $x\in[a,b]$, we have*

$$\left|f(x) - \frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{1.9}$$
$$\leq \frac{1}{4\sqrt{3}}(b-a)(\Phi-\varphi)\,.$$

In [3], Bernett et al., by the use of Chebeyshev's functional, improved the Matic et al. result by providing first membership of the right side of (1.9) in terms of Euclidean norm as follows:

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous mapping whose derivative $f'\in L_2[a,b]$, then we have,*

$$\begin{aligned}
&\left| f(x) - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\
&\leq \frac{(b-a)}{2\sqrt{3}}\left(\frac{1}{(b-a)}\left\|f'\right\|_2^2 - \left(\frac{f(b)-f(a)}{(b-a)}\right)^2\right)^{\frac{1}{2}} \\
&\leq \frac{(b-a)(\Phi-\varphi)}{4\sqrt{3}}, \text{ if } \varphi \leq f'(x) \leq \Phi \text{ for a.e.t on } [a,b],
\end{aligned}$$

for all $x \in [a,b]$.

During the past few years many researchers have given considerable attention to the above inequalities and various generalizations, extensions and variants of these inequalities have appeared in the literature, see [1]-[3], [6], [19] and the references cited therein. For recent results and generalizations concerning Ostrowski and Grüss inequalities, we refer the reader to the recent papers [1]-[3], [6], [17]-[19].

The theory of fractional calculus has known an intensive development over the last few decades. It is shown that derivatives and integrals of fractional type provide an adequate mathematical modelling of real objects and processes see ([7]-[13]). Therefore, the study of fractional differential equations need more developmental of inequalities of fractional type. The main aim of this work is to develop new integral inequalities of Ostrowski-Grüss type for Riemann-Liouville fractional integrals. From our results, the classical Ostrowski-Grüss type inequalities can be deduced as some special cases. Let us begin by introducing this type of inequality.

In [7] and [11], the authors established some inequalities for differentiable mappings which are connected with Ostrowski type inequality by used the Riemann-Liouville fractional integrals, and they used the following lemma to prove their results:

Lemma 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable function on I° with $a, b \in I$ $(a < b)$ and $f' \in L_1[a,b]$, then*

(1.10)

$$f(x) = \frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha} J_a^\alpha f(b) - J_a^{\alpha-1}(P_2(x,b)f(b)) + J_a^\alpha(P_2(x,b)f^{'}(b)), \quad \alpha \geq 1,$$

where $P_2(x,t)$ is the fractional Peano kernel defined by

$$P_2(x,t) = \begin{cases} \left(\dfrac{t-a}{b-a}\right)(b-x)^{1-\alpha}\Gamma(\alpha), & a \leq t < x \\ \\ \left(\dfrac{t-b}{b-a}\right)(b-x)^{1-\alpha}\Gamma(\alpha), & x \leq t \leq b. \end{cases} \tag{1.11}$$

In [7] and [11], the authors derive the following interesting fractional integral inequality:

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}(b-x)^{1-\alpha}\Gamma(\alpha) J_a^\alpha(f(b)) + J_a^{\alpha-1}(P_2(x,b)f(b)) \right| \\
&\leq \frac{M}{\alpha(\alpha+1)}\left[(b-x)\left(2\alpha\left(\frac{b-x}{b-a}\right) - \alpha - 1\right) + (b-a)^\alpha (b-x)^{1-\alpha}\right]
\end{aligned}$$

under the assumption that $|f'(x)| \leq M$, for any $x \in [a,b]$.

Firstly, we give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. More details, one can consult [14] and [15].

Definition 1. *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ with $a \geq 0$ is defined as*

$$\begin{aligned} J_a^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \\ J_a^0 f(x) &= f(x). \end{aligned}$$

Recently, many authors have studied a number of inequalities by used the Riemann-Liouville fractional integrals, see ([7]-[13]) and the references cited therein.

2. Main Results

Theorem 4. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping in I^0 (interior of I), and $a, b \in I^0$ with $a < b$ and $f' \in L_2[a,b]$. If $f^{'} : (a,b) \to \mathbb{R}$ is bounded on (a,b) with $\varphi \leq f'(x) \leq \Phi$, then we have*

$$\begin{aligned} &\left| \frac{f(x)}{\Gamma(\alpha)} - \frac{(b-x)^{1-\alpha}}{(b-a)} J_a^\alpha f(b) + \frac{1}{\Gamma(\alpha)} J_a^{\alpha-1} \left(P(x,b) f(b)\right) \right. \\ &\left. - \left(\frac{f(b)-f(a)}{b-a} \right) \times \left(\frac{(b-x)^{1-\alpha}(b-a)^\alpha}{\Gamma(\alpha+2)} - \frac{(b-x)}{\Gamma(\alpha+1)} \right) \right| \\ &\leq (b-a)(K(x))^{\frac{1}{2}} \left(\frac{1}{(b-a)\Gamma^2(\alpha)} \left\| f^{'} \right\|_2^2 - \left(\frac{f(b)-f(a)}{(b-a)\Gamma(\alpha)} \right)^2 \right)^{\frac{1}{2}} \\ &\leq \frac{(K(x))^{\frac{1}{2}}}{2\Gamma(\alpha)} (b-a)(\Phi - \varphi) \end{aligned} \tag{2.1}$$

for all $x \in [a,b]$ and $\alpha \geq 1$ where

$$\begin{aligned} K(x) &= (b-x)^{1-\alpha}(b-a)^{2\alpha-2} \left(\frac{1}{2\alpha+1} + \frac{1}{2\alpha-1} - \frac{1}{\alpha} \right) \\ &\quad + \frac{(b-x)^\alpha}{(b-a)^2} \left(\frac{b-x}{\alpha} - \frac{b-a}{2\alpha-1} \right) - \left(\frac{(b-x)^{1-\alpha}(b-a)^{\alpha-1}}{\alpha(\alpha+1)} - \frac{(b-x)}{\alpha(b-a)} \right)^2. \end{aligned}$$

Proof. We consider the fractional Peano kernel $P_2 : [a,b]^2 \to \mathbb{R}$ as defined in (1.11). Using Korkine's identity

$$T(f,g) := \frac{1}{2(b-a)^2} \int_a^b \int_a^b (f(t)-f(s))(g(t)-g(s))\, dsdt$$

we obtain

$$\begin{aligned}
&\frac{1}{(b-a)\Gamma^2(\alpha)}\int_a^b (b-t)^{\alpha-1}P_2(x,t)f'(t)\,dt \\
&-\left(\frac{1}{(b-a)\Gamma(\alpha)}\int_a^b (b-t)^{\alpha-1}P_2(x,t)\,dt\right)\left(\frac{1}{(b-a)\Gamma(\alpha)}\int_a^b f'(t)\,dt\right) \\
=\;&\frac{1}{2(b-a)^2\Gamma^2(\alpha)}\int_a^b\int_a^b\left[(b-t)^{\alpha-1}P_2(x,t)-(b-s)^{\alpha-1}P_2(x,s)\right]\left[f'(t)-f'(s)\right]dsdt
\end{aligned}$$

i.e.

$$\begin{aligned}
&\frac{1}{(b-a)\Gamma(\alpha)}J_a^\alpha\left(P_2(x,b)f'(b)\right)-\frac{1}{(b-a)\Gamma(\alpha)}J_a^\alpha\left(P_2(x,b)\right)\left(\frac{f(b)-f(a)}{b-a}\right) \\
=\;&\frac{1}{2(b-a)^2\Gamma^2(\alpha)}\int_a^b\int_a^b\left[(b-t)^{\alpha-1}P_2(x,t)-(b-s)^{\alpha-1}P_2(x,s)\right]\left[f'(t)-f'(s)\right]dsdt.
\end{aligned}\tag{2.2}$$

Since

$$J_a^\alpha\left(P_2(x,b)f'(b)\right)=f(x)-\frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha}J_a^\alpha f(b)+J_a^{\alpha-1}\left(P_2(x,b)f(b)\right)\tag{2.3}$$

and

$$J_a^\alpha\left(P_2(x,b)\right)=\frac{(b-x)^{1-\alpha}(b-a)^\alpha}{\alpha(\alpha+1)}-\frac{(b-x)}{\alpha},\tag{2.4}$$

then by (2.2) we get the following identity,

$$\begin{aligned}
&\frac{f(x)}{(b-a)\Gamma(\alpha)}-\frac{(b-x)^{1-\alpha}}{(b-a)^2}J_a^\alpha f(b)+\frac{1}{(b-a)\Gamma(\alpha)}J_a^{\alpha-1}\left(P(x,b)f(b)\right) \\
&-\frac{1}{(b-a)}\left(\frac{f(b)-f(a)}{b-a}\right)\times\left(\frac{(b-x)^{1-\alpha}(b-a)^\alpha}{\Gamma(\alpha+2)}-\frac{(b-x)}{\Gamma(\alpha+1)}\right) \\
=\;&\frac{1}{2(b-a)^2\Gamma^2(\alpha)}\int_a^b\int_a^b\left[(b-t)^{\alpha-1}P_2(x,t)-(b-s)^{\alpha-1}P_2(x,s)\right]\left[f'(t)-f'(s)\right]dsdt.
\end{aligned}\tag{2.5}$$

Using the Cauchy-Swartz inequality for double integrals, we write

$$\begin{aligned}
&\frac{1}{2\left(b-a\right)^{2}\Gamma^{2}(\alpha)}\left|\int_{a}^{b}\int_{a}^{b}\left[\left(b-t\right)^{\alpha-1}P_{2}\left(x,t\right)-\left(b-s\right)^{\alpha-1}P_{2}\left(x,s\right)\right]\left[f^{'}\left(t\right)-f^{'}\left(s\right)\right]dsdt\right| \\
\leq &\left(\frac{1}{2\left(b-a\right)^{2}\Gamma^{2}(\alpha)}\int_{a}^{b}\int_{a}^{b}\left[\left(b-t\right)^{\alpha-1}P_{2}\left(x,t\right)-\left(b-s\right)^{\alpha-1}P_{2}\left(x,s\right)\right]^{2}dsdt\right)^{\frac{1}{2}} \\
&\times\left(\frac{1}{2\left(b-a\right)^{2}\Gamma^{2}(\alpha)}\int_{a}^{b}\int_{a}^{b}\left[f^{'}\left(t\right)-f^{'}\left(s\right)\right]^{2}dsdt\right)^{\frac{1}{2}}.
\end{aligned}\tag{2.6}$$

However,

$$\begin{aligned}
&\frac{1}{2\left(b-a\right)^{2}\Gamma^{2}(\alpha)}\int_{a}^{b}\int_{a}^{b}\left[\left(b-t\right)^{\alpha-1}P_{2}\left(x,t\right)-\left(b-s\right)^{\alpha-1}P_{2}\left(x,s\right)\right]^{2}dsdt \\
= &\frac{1}{\left(b-a\right)\Gamma^{2}(\alpha)}\int_{a}^{b}\left(b-t\right)^{2\alpha-2}P_{2}^{2}\left(x,t\right)dt-\left(\frac{1}{\left(b-a\right)\Gamma(\alpha)}\int_{a}^{b}\left(b-t\right)^{\alpha-1}P_{2}\left(x,t\right)dt\right)^{2} \\
= &\left(b-x\right)^{1-\alpha}\left(b-a\right)^{2\alpha-2}\left(\frac{1}{2\alpha+1}+\frac{1}{2\alpha-1}-\frac{1}{\alpha}\right)+\frac{\left(b-x\right)^{\alpha}}{\left(b-a\right)^{2}}\left(\frac{b-x}{\alpha}-\frac{b-a}{2\alpha-1}\right) \\
&-\left(\frac{\left(b-x\right)^{1-\alpha}\left(b-a\right)^{\alpha-1}}{\alpha\left(\alpha+1\right)}-\frac{\left(b-x\right)}{\alpha\left(b-a\right)}\right)^{2},
\end{aligned}\tag{2.7}$$

and

$$\frac{1}{2\left(b-a\right)^{2}\Gamma^{2}\left(\alpha\right)}\int_{a}^{b}\int_{a}^{b}\left(f^{'}\left(t\right)-f^{'}\left(s\right)\right)^{2}dsdt=\frac{1}{(b-a)\Gamma^{2}\left(\alpha\right)}\left\Vert f^{'}\right\Vert_{2}^{2}-\left(\frac{f\left(b\right)-f\left(a\right)}{(b-a)\Gamma\left(\alpha\right)}\right)^{2}.\tag{2.8}$$

Using (2.6)-(2.8), we deduce the (2.2) inequality.

Moreover, if $\varphi\leq f'(t)\leq\Phi$ almost everywhere t on (a,b), then by using Grüss inequality, we get

$$0\leq\frac{1}{b-a}\int_{a}^{b}\left(f'\left(t\right)\right)^{2}dt-\left(\frac{1}{b-a}\int_{a}^{b}f'\left(t\right)dt\right)^{2}\leq\frac{1}{4}(\Phi-\varphi)^{2},$$

which proves the last inequality of (2.1). □

Corollary 1. *Under the assumptions of Theorem 4 with $\alpha = 1$. Then the following inequality holds:For*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right)\right| \tag{2.9}$$

$$\leq \frac{(b-a)}{2\sqrt{3}}\left(\frac{1}{(b-a)}\left\|f'\right\|_2^2 - \left(\frac{f(b)-f(a)}{(b-a)}\right)^2\right)^{\frac{1}{2}}$$

$$\leq \frac{(b-a)(\Phi-\varphi)}{4\sqrt{3}}.$$

Proof. Proof of Corollary 1 can be as similar to the proof of Theorem 4. □

Remark 1. *If we take $x = \frac{a+b}{2}$ in (2.9), it follows that*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{(b-a)}{2\sqrt{3}}\left(\frac{1}{(b-a)}\left\|f'\right\|_2^2 - \left(\frac{f(b)-f(a)}{(b-a)}\right)^2\right)^{\frac{1}{2}}$$

$$\leq \frac{(b-a)(\Phi-\varphi)}{4\sqrt{3}}.$$

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